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Space-Time Quantization and Nonlocal Field Theory

—Relativistic Second Quantization of Matrix Model—

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ABSTRACT

We propose relativistic second quantization of matrix model of D particles in a general framework of nonlocal field theory based on Snyder-Yang's quantized space-time. Second-quantized nonlocal field is in general noncommutative with quantized space-time, but conjectured to become commutative with light cone time X^+. This conjecture enables us to find second-quantized Hamiltonian of D particle system and Heisenberg's equation of motion of second-quantized **D** field in close contact with Hamiltonian given in matrix model. We propose Hamilton's principle of Lorentz-invariant action of **D** field and investigate what conditions or approximations are needed to reproduce the above Heisenberg's equation given in light cone time. Both noncommutativities appearing in position coordinates of D particles in matrix model and in quantized space-time will be eventually unified through second quantization of matrix model.

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1. Introduction

In the previous paper[1] (referred to as I, hereafter) the present author has reexamined the original idea of nonlocal field and clarified the necessity of introducing the concept of noncommutative space-time for the comprehensive description of extended objects. In fact, in the original bilocal field theories[2],[3] nonlocal field U was characterized by noncommutativity with space-time coordinates X_μ, that is,

$$[U, X_\mu] \neq 0, \tag{1.1}$$

discarding the restriction that field quantities are simply functions of space-time coordinates, but retaining the classical nature of the latter space-time coordinates. Consequently, as was shown by Yukawa[3], if one chooses the representation basis vector, $|x_\mu\rangle$, with simultaneous eigenvalues x_μ of all the space-time coordinate components X_μ , let us call it the space-time coordinate representation hereafter, one immediately arrives at bilocal field $\langle x'_\mu|U|x''_\mu\rangle \equiv U(x'_\mu, x''_\mu)$, i.e., a simple two-point function of space-time coordinates, which is clearly insufficient to describe extended objects in general.

This consideration naturally leads us to the idea of the noncommutative space-time, which directly invalidates the use of the above space-time coordinate representation and enables us to remove the limitation on the concept of nonlocal fields mentioned above. It should be further noticed that under the idea of noncommutative space-time, the original assumption (1.1) itself becomes a natural consequence of the idea, because nonlocal field U, even if it is naively a function of the noncommutative space-time, can never be commutative with space-time coordinates.

As was pointed out in I, the idea of noncommutative space-time was early proposed by Snyder[4] and Yang[5] in terms of "quantized space-time" in a Lorentz-covariant way. According to their idea, we tried in I to describe both nonlocal field U and noncommutative space-time quantities as operators working on the common linear space of a certain infinite series of functions of Snyder-Yang's parameters and to represent them in terms of noncommutative infinite-dimensional matrices.

In the present paper, we wish to extend this idea and apply it to the recent matrix model. This model is expected to describe quantum-mechanical many-body system of D-0 brane or D-particle, a typical extended object which is nowadays widely believed to be fundamental constituent of superstrings.[6] As is well known, the idea of matrix model of position coordinates of D-branes actually motivated the concept of noncommutative space-time[7] and the recent remarkable studies on noncommutative geometry.[8] In the present paper, however, we take a viewpoint slightly different from the latter approach, and propose Lorentz-invariant second-quantized field theory of matrix model on the basis of noncommutative and quantized space-time of Snyder-Yang's type from the beginning. It turns out that second quantization of matrix model as quantum mechanical system of many D-particles is crucial in the Lorentz-covariant formulation of the latter model, the importance of which has been emphasized so far by several authors.[6][9]

The plan of the present paper is as follows. In section 2, first we briefly recapitulate the quantized space-time algebra $\mathcal{R}$ originally proposed by Snyder[4] and Yang.[5] We introduce Hilbert space I, on which quantized space-time quantities $\mathcal{R}$ work as operators. In section 3, nonlocal field U is defined as an operator working on Hilbert space I on an equal footing with $\mathcal{R}$. The light cone representation bases are introduced in Hilbert space I, where both of nonlocal field U and light cone time operator X^+ are assumed to become diagonal. This framework enables us to construct second-quantized theory of nonlocal field $\mathbf{U}$ according to Hamiltonian formalism in Hilbert space II as in the usual quantum field theory. In section 4, the idea is applied to the infinite number of D particle system described by matrix model, whose Hamiltonian is now given in terms of second-quantized $\mathbf{D}$ field in light cone representation bases. In section 5, we propose Lorentz invariant action of $\mathbf{D}$ field and investigate what conditions or approximations are needed to reproduce Heisenberg's equation of motion given in section 4. The final section is devoted to conclusions and discussions.

2. Recapitulation of Quantized Space-time and Quasi-local Representation

As was emphasized in I, Snyder[4] first challenged the idea of quantized space-time, by expressing space-time quantities by linear differential operators on the five-dimensional background space. Later on, Yang[5] developed the idea by extending the background space to the six-dimensional space $(\xi_0, \xi_1, \xi_2, \xi_3, \eta, \tau)$, where both space-time coordinate X_μ and momentum (or translation) P_μ operators were defined as linear differential operators on the space:

$$X_i = i(\xi_i \frac{\partial}{\partial \eta} - \eta \frac{\partial}{\partial \xi_i}), \tag{2.1}$$

$$X_0 = i(\xi_0 \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial \xi_0}) \tag{2.2}$$

and

$$P_i = -i(\xi_i \frac{\partial}{\partial \tau} - \tau \frac{\partial}{\partial \xi_i}), \tag{2.3}$$

$$P_0 = -i(\xi_0 \frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial \xi_0}). \tag{2.4}$$

We call hereafter the space $(\xi_0, \xi_1, \xi_2, \xi_3, \eta, \tau)$ and its dimensional extension in general, Snyder-Yang's parameter space.

Now let us recapitulate, for the later convenience, the typical commutation relations between the above quantities characteristic of the noncommutative space-time:

$$[X_i, X_j] = -iL_{ij} \tag{2.5}$$

$$[X_i, X_0] = iM_i, \tag{2.6}$$

and

$$[P_i, P_j] = -iL_{ij} \tag{2.7}$$

$$[P_i, P_0] = iM_i, \tag{2.8}$$

with L_{ij} and M_i defined by

$$L_{ij} = i(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}) \tag{2.9}$$

and

$$M_i = i(\xi_0 \frac{\partial}{\partial \xi_i} + \xi_i \frac{\partial}{\partial \xi_0}). \tag{2.10}$$

L_{ij} and M_i as a whole constitute six Lorentz generators, satisfying the well-known Lorentz algebra, under which $X_{i,0}$ and $P_{i,0}$ are transformed as Lorentz four-vectors. Furthermore,

$$[X_i, P_j] = i\delta_{ij} N, \tag{2.11}$$

$$[X_0, P_0] = -iN, \tag{2.12}$$

$$[X_i, P_0] = [P_i, X_0] = 0. \tag{2.13}$$

Here one finds the operator N

$$N = i(\eta \frac{\partial}{\partial \tau} - \tau \frac{\partial}{\partial \eta}) \tag{2.14}$$

is clearly Lorentz-scalar and plays the role of the usual quantum mechanical Planck constant h, and further it causes the reciprocal transformation between the coordinates and momenta as

$$[X_i, N] = -iP_i, \cdots \tag{2.15}$$

It is important to note that the fifteen operators introduced above

$$\mathcal{R}_{15} \equiv (X_\mu, P_\mu, L_{\mu\nu}, N) \tag{2.16}$$

with $\mu = (i, 0)$ and $L_{\mu\nu} \equiv (L_{ij}, M_i)$, constitute as a whole a Lie algebra, which is interpreted as a maximal set of Killing vector fields realized on a pseudosphere in six-dimensional Minkowski-space; $-{\xi_0}^2 + {\xi_1}^2 + {\xi_2}^2 + {\xi_3}^2 + \eta^2 + \tau^2 = constant.$

The original Snyder's algebra is now understood as one missing τ in the above Yang's model. Four-momenta P_μ are there defined as $P_i = \xi_i/\eta, P_0 = \xi_0/\eta$, while space-time coordinate operators being the same as (2.1) and (2.2). They show, therefore, different commutation relations from Yang's ones; for instance, $[X_i, P_j] = -iP_iP_j, [X_0, P_i] = -iP_0P_i$. In this case, $\mathcal{R}_{10} \equiv (X_\mu, L_{\mu\nu})$ constitute a Lie algebra or a maximal set of Killing vector fields realized on a pseudosphere in five-dimensional Minkowski-space; $-{\xi_0}^2 + {\xi_1}^2 + {\xi_2}^2 + {\xi_3}^2 + \eta^2 = constant$. In what follows, let us call Lie algebra $\mathcal{R}$ constituted by Killing vector fields in general the space-time algebra of Snyder-Yang's type.

Among $\mathcal{R}_{15}$ in Yang's model, (2.16), which will be preferably adopted in what follows, one finds that the time-like operators X_0, P_0, and M_i have continuous eigenvalues, contrary to those of the space-like operators X_i, P_i, L_{ij} and N, which have discrete eigenvalues. This statement is easily confirmed from the following fact that, for instance, X_i and X_0 are rewritten as

$$\begin{aligned} X_i &= \frac{1}{i}\frac{\partial}{\partial \alpha_i}, \\ X_0 &= \frac{1}{i}\frac{\partial}{\partial \alpha_0} \end{aligned} \tag{2.17}$$

under fixed $k_i(= \sqrt{\eta^2 + {\xi_i}^2})$ and $k_0(= \sqrt{\eta^2 - {\xi_0}^2})$, respectively, where $\xi_i = k_i \sin\alpha_i, \eta = k_i \cos\alpha_i$ and $\xi_0 = k_0 \sinh\alpha_0$, $\eta = k_0 \cosh\alpha_0$. Eigenfunctions of X_i and X_0 are given by $\exp(in_i\alpha_i)$ and $\exp(it\alpha_0)$ with eigenvalues of integer n_i and continuous time t, respectively.

By using the above result, let us here consider a representation basis of the space-time algebra $\mathcal{R}$, i.e., quasi-local space-time bases, Q_n :

$$Q_n = \prod_i \{\exp in_i\alpha_i\}\{\exp it\alpha_0\} f_n(-{\xi_0}^2 + {\xi_1}^2 + {\xi_2}^2 + {\xi_3}^2 + \eta^2, \tau), \tag{2.18}$$

where i runs from 1 to 3. At this point, one finds that each exponential factor on the right-hand side is, respectively, eigenstate of i-th space coordinate X_i with

discrete eigenvalue n_i and X_0 with continuous eigenvalue t, but basis function Q_n itself being a product of these factors can no longer be eigenfunction of neither of $X_i's$ and X_0 on account of their noncommutativity, while the last factor f_n being ortho-normal function of $(-{\xi_0}^2+{\xi_1}^2+{\xi_2}^2+{\xi_3}^2+\eta^2)$ and τ is clearly eigenfunction of each X_i and X_0 with a vanishing value.

Furthermore, one sees the following basic relations

$$X_i Q_n = n_i Q_n + \sum_{j\neq i} \frac{n_j \xi_j \xi_i}{(\eta^2 + {\xi_j}^2)} Q_n + \frac{t \xi_0 \xi_i}{(\eta^2 - \xi_0^2)} Q_n \tag{2.19}$$

and

$$X_0 Q_n = t Q_n + \sum_i \frac{n_i \xi_i \xi_0}{(\eta^2 + {\xi_i}^2)}. \tag{2.20}$$

From (2.19) and (2.20), one obtains the following important result that the expectation values of each X_i and X_0 remain n_i and t;

$$\bar{X}_i \equiv \langle Q_n | X_i | Q_n \rangle = n_i \tag{2.21}$$

with fluctuation $\Delta {X_i}^{(n)}$

$$\begin{aligned}(\Delta {X_i}^{(n)})^2 &\equiv \langle Q_n | (X_i - n_i)^2 | Q_n \rangle \\ &= \sum_{j\neq i} {n_j}^2 \langle Q_n | \frac{{\xi_j}^2 {\xi_i}^2}{(\eta^2 + \xi_j^2)^2} | Q_n \rangle + t^2 \langle Q_n | \frac{{\xi_0}^2 {\xi_i}^2}{(\eta^2 - {\xi_0}^2)^2} | Q_n \rangle,\end{aligned} \tag{2.22}$$

and

$$\bar{X}_0 \equiv \langle Q_n | X_0 | Q_n \rangle = t \tag{2.23}$$

with fluctuation $\Delta {X_0}^{(n)}$

$$\begin{aligned}(\Delta {X_0}^{(n)})^2 &\equiv \langle Q_n | (X_0 - t)^2 | Q_n \rangle \\ &= \sum_i {n_i}^2 \langle Q_n | \frac{{\xi_i}^2 {\xi_0}^2}{(\eta^2 + \xi_i^2)^2} | Q_n \rangle.\end{aligned} \tag{2.24}$$

In the preceding argument, one can easily generalize the spatial dimension 3,

tacitly assumed in the above arguments, to the arbitrary dimension by extending the dimension of Snyder-Yang's parameter space as $(\xi_0, \xi_1, \xi_2, \cdots, \xi_p; \eta, \tau)$.

In what follows, we call the representation space spanned by the basis vectors $|Q_n\rangle's$ discussed in this section, Hilbert space I, in contrast to Hilbert Space II which will be introduced in section 4 in the usual quantum field-theoretical sense. It should be noted that the present quasi-local representation functions Q_n in Hilbert space I possibly describe a four-dimensional excitation state quasi-localized in space and time directions, and enable us to find correspondence to a conventional local field, as will be discussed in the last section.

3. Matrix Representation on the light cone bases and Nonlocal Field

As was stated in Introduction, nonlocal field U also is defined as an operator on Hilbert space I on an equal footing with $\mathcal{R}$ discussed in the preceding section. In this connection, it is quite important to notice that the representation basis functions such as Q_n's, on which noncommutative algebra $\mathcal{R}$ and nonlocal field U work as operators, can never be defined on a definite time value in general. One therefore encounters the entirely different situation from the familiar quantum mechanics, where any complete set of representation bases is defined on a fixed time.

At this point, it is quite instructive to remember that Dirac[10] early pointed out the importance of a role of light cone time in terms of "front" in his systematic consideration on "Forms of Relativistic Dynamics". More recently, in the study of string field theories, light cone variables $X^{\pm}$ play an important role to describe the string as an extended object in space and time. Indeed, Kaku and Kikkawa[11] showed in their first formulation of quantum field theory of string, that string fields and their interactions can be described on a definite light cone time $X^+ = \tau$ under the so-called light cone gauge fixing. Furthermore, it is interesting to note that the recent matrix model[6] or quantum mechanics of D particles as

fundamental constituents of superstrings is also successfully formulated in the "infinite momentum frame." In accordance with these thoughts, it seems promising to conjecture that any nonlocal field which is nonlocal in space-time in general, becomes local with respect to time in the infinite momentum frame so as to allow to describe its time development.

This conjecture comes also from a simple consideration of special theory of relativity that any region in space-time in the usual Lorentz frame with a certain finite time width or time uncertainty ($\Delta X_0 = \epsilon$) may be seen in the infinite momentum frame, as a region with a vanishingly small width of light cone time, $\Delta X^+ = 0$. This fact strongly supports the view that any nonlocal field U becomes to have a definite light cone time in the infinite momentum frame.

In what follows, let us consider this possibility in the present noncommutative version of space-time, by assuming the following commutativity in general⋆

$$[X^+, U] = 0, \tag{3.1}$$

where $X^\pm \equiv \frac{1}{\sqrt{2}}(X_0 \pm X_3)$ in the present four-dimensional space-time.

According to this assumption, let us now take the following *light cone representation* bases $F_n(t)'s$, in which both X^+ and U are diagonal, with eigenvalues of continuous t and $u_n(t)$, respectively, that is,

$$X^+ F_n(t) (\equiv i(\xi^+ \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial \xi^-}) F_n(\xi_i, \xi^\pm, \eta, \tau; t)) = t F_n(t) \tag{3.2}$$

and

$$U = \sum_{n,t} |F_n(t)\rangle u_n(t) \langle F_n(t)|, \tag{3.3}$$

where $\xi^\pm \equiv \frac{1}{\sqrt{2}}(\xi_0 \pm \xi_3)$ and $|F_n(t)\rangle$ being ket vector corresponding to the basis function $F_n(t) \equiv F_n(\xi_i, \xi^\pm, \eta, \tau; t)$. In the above expression and hereafter, we tacitly use such a notation $\sum_t$ even for continuous parameter t in place of $\int dt$.

⋆ It should be noted that any light cone time X^+ defined in each Lorentz frame boosted in a certain spatial direction is proportional to time in the infinite momentum frame realized in the same direction.

By remarking the expression $X^{\pm} = i(\xi^{\pm}\frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial\xi^{\mp}})$ and

$$X^{\pm} \exp\,(\frac{-it\eta}{\xi^{\pm}}) = t\exp(\frac{-it\eta}{\xi^{\pm}}), \tag{3.4}$$

one finds $F_n(t)$ satisfying (3.2) is actually given by

$$F_n(t) = \exp(\frac{-it\eta}{\xi^{+}})g_n(\xi_i, \xi^{+}, \tau; t) \tag{3.5}$$

with suitable ortho-normalized functions g_n.

At this point, it should be noted that any space-time operator R belonging to $\mathcal{R}$ other than X^{+}, must be represented in the following non-diagonal form in general

$$R = \sum_{mn,tt'} |F_m(t)\rangle\langle F_m(t)|R|F_n(t')\rangle\langle F_n(t')|. \tag{3.6}$$

4. Second Quantization and Hamiltonian Formalism in Matrix Model of D-particles

Now let us turn our attention to the fundamental problem how to determine the time development of nonlocal field U, or the time development of $u_n(t)$ given in (3.3). According to the ordinary idea of quantum mechanics, we expect this to be done by means of the Hamiltonian formalism. Indeed we find an important clue for this in matrix model[6] of D-particles formulated in the infinite momentum frame or light cone time as mentioned in the preceding section. Let us call our attention to the Hamiltonian presented in the latter theory:

$$H = C\,\mathrm{tr}\,\{\frac{P_iP_i}{2} + \frac{1}{4}[X_i, X_j]^2\}, \tag{4.1}$$

where we neglected the terms of supersymmetric partner. Here, X_i and P_i ($i = 1, 2, 3, \cdots, 9$), on which tr works, are $N \times N$ matrices and interpreted to express

transverse components of position coordinates and momenta of N D-particles in the infinite momentum frame realized in the 11-th spatial direction in 11 dimensional space-time. Especially the diagonal matrix elements $\langle n|X_i|n\rangle (n = 1, 2, \cdots, N)$ are understood as denoting literally *position coordinates*[7] of the n-th D-particle.

At first sight, these $N \times N$ matrices X_i and P_i noncommutative in general may be identifiable in the N infinite limit with our space-time quantities X_i and P_i belonging to space-time algebra $\mathcal{R}$ which are also expressed in noncommutative matrix form, $\langle F_m(t)|R|F_n(t')\rangle$ as seen in (3.6). However, it should be emphasized that the former $N \times N$ matrices are interpreted to express position coordinates and momenta of N D-particles, contrary to the latter ones which are indeed concerned with space-time quantities $\mathcal{R}$ *irrelevant* to individual objects involved in space-time. One remembers a similar situation in the familiar second quantization procedure in quantum field theory, where a whole set of individual *position coordinates* of identical particles finally turns into *spatial coordinates* as arguments of quantized field operators.

Bearing this point in mind, let us first try to carry out second quantization of identical D-particle system by introducing the second-quantized D-particle field, $\mathbf{D}$, on the light cone basis according to (3.3):

$$\mathbf{D} = \sum_{n,t} |F_n(t)\rangle \mathbf{d}_n(t) \langle F_n(t)|, \tag{4.2}$$

where light cone basis function $F_n(t)$ is defined by (3.5), simply extending the space-time dimension from 4 to 11. $\mathbf{d}_n(t)$ and its adjoint $\mathbf{d}_n(t)^\dagger$ denote second-quantized operators satisfying the following commutation relations:

$$[\mathbf{d}_m(t), \mathbf{d}_n(t)^\dagger] = \delta_{mn} \tag{4.3}$$

and thus

$$[\mathbf{D}, \mathbf{D}^\dagger] = \mathbf{1}. \tag{4.4}$$

Here one notices that D-particles are assumed to be identical particles obeying

Bose-Einstein statistics, and that $\mathbf{d}_n$ and its adjoint ${\mathbf{d}_n}^\dagger$ are, respectively, annihilation and creation operators of excitation mode, $|F_n\rangle$ of D-field. In what follows, let us call the Hilbert space, on which the above commutation relations are realized, Hilbert space II by distinguishing it from Hilbert space I spanned by $|F_n(t)\rangle's$ or $|Q_n\rangle's$. According to this definition, one finds that the second quantized D-field $\mathbf{D}$ is an operator both in Hilbert space I and II.

Now let us try to construct second-quantized Hamiltonian on a definite light cone time t in terms of the $\mathbf{D}$ field defined in Hilbert space I, II and space-time quantities, X_i and P_i defined in Hilbert space I, in close contact with Hamiltonian H in (4.1). At this point, let us regard the second term*on the right-hand side of (4.1) as the so-called two-particle term which expresses the interactions between two D-particles, in contrast to the first term expressing one-particle term concerning N individual D-particles. This consideration naturally leads us to the following form of second-quantized Hamiltonian $\mathbf{H}(t)$ defined on light cone time t as an operator in Hilbert space II, which corresponds to H in (4.1) of quantum mechanical many- D-particle system,

$$
\begin{aligned}
\mathbf{H}(t) &= \frac{C}{2}\,\mathrm{Tr}_t\,\{\,\mathbf{D}^\dagger P_i P_i \mathbf{D}\} + \frac{C}{4}\,\mathrm{Tr}_t\,\{\,\mathbf{D}^\dagger [X_i,X_j]\mathbf{D}\,\mathbf{D}^\dagger [X_i,X_j]\mathbf{D}\} \\
&= \frac{C}{2}\sum_n \langle F_n(t)|P_iP_i|F_n(t)\rangle \mathbf{d}_n(t)^\dagger \mathbf{d}_n(t) \\
&+ \frac{C}{4}\sum_{mn} \langle F_n(t)|[X_i,X_j]|F_m(t)\rangle\langle F_m(t)|[X_i,X_j]|F_n(t)\rangle \\
&\qquad \times \mathbf{d}_n(t)^\dagger \mathbf{d}_m(t)\mathbf{d}_m(t)^\dagger \mathbf{d}_n(t),
\end{aligned}
\tag{4.5}
$$

where Tr_t means trace concerned with sub-space of Hilbert space I spanned by basis vectors $|F_n(t)\rangle's$ with fixed time t. In the above derivation of the second expression in (4.5), it is crucial that $L_{ij} \equiv i[X_i,X_j]$ defined in (2.5) is commutative with X^+, in restricting intermediate states to be on fixed light cone time t.

$\star$ One can also regard this term as a kind of self-energy, which is neglected in the present paper.

In order to examine directly the relation between H in (4.1), i.e., quantum-mechanical N-body Hamiltonian of D-particles in matrix model and the present quantum field-theoretic one $\mathbf{H}(t)$ given in (4.5), let us first define quantum field-theoretic N-body state at $t = 0$ in terms of $\mathbf{d}_n (\equiv \mathbf{d}_n(0))$ as

$$|N_1 N_2 \ldots ; N\rangle\rangle \equiv A{\mathbf{d}_1{}^\dagger}^{N_1} {\mathbf{d}_2{}^\dagger}^{N_2} \ldots |0\rangle\rangle, \tag{4.6}$$

with $N_1 + N_2 + \ldots = N$, where N_n denotes eigenvalue of number operator $\mathbf{N}_n \equiv \mathbf{d}_n{}^\dagger \mathbf{d}_n$, A is a normalization constant and $|0\rangle\rangle$ is the quantum field theoretic vacuum state of D-particle, that is, the ground state in Hilbert space II. Now it is easy to calculate the expectation value of $\mathbf{H}(0)$ with respect to this N-body state with the result:

$$\begin{aligned} &\langle\langle N_1 N_2 \ldots ; N | \mathbf{H} | N_1 N_2 \ldots ; N \rangle\rangle \\ &= \frac{C}{2} \sum_n N_n \langle F_n | \frac{P_i P_i}{2} | F_n \rangle + \frac{C}{4} \sum_{mn} N_m N_n \langle F_m | [X_i, X_j] | F_n \rangle \langle F_n | [X_i, X_j] | F_m \rangle \end{aligned} \tag{4.7}$$

with $|F_n\rangle$ being $|F_n(t = 0)\rangle$, where we omitted the self-contraction term arising in the second term in the last expression.

At this point, let us set all of N_n to be equal to unity, then (4.7) becomes

$$C \sum_n \langle F_n | \{ \frac{P_i P_i}{2} + \frac{1}{4} [X_i, X_j]^2 \} | F_n \rangle. \tag{4.8}$$

One should remark here that this expression formally coincides with the expression of H in (4.1), that is, the quantum mechanical Hamiltonian of N-body D-particle system with infinite N limit, if one identifies the infinite dimensional matrix elements $\langle m | R | n \rangle$ in matrix model, where $m(n)$ is interpreted to mean the $m(n)$-th D-particle, with the corresponding quantities $\langle F_m | R | F_n \rangle$, which are defined in Hilbert space I and interpreted as concerned with $m(n)$-th *excitation mode* of D-field. This result strongly supports our choice of quantum field theoretic Hamiltonian of D-particles presented in (4.5), if one takes into account that

quantum mechanical N D-particles are *identical particles* indistinguishable from each other, by virtue of the Weyl group of $U(N)$.[7]

Now we are in a position to consider the time development of D field, that is, Heisenberg's equation of motion. By noticing again $L_{ij} = i[X_i, X_j]$ defined in (2.5), it can be written down as follows

$$
\begin{aligned}
i\hbar \frac{d}{dt}\mathbf{d}_n(t) &= [\mathbf{d}_n(t), \mathbf{H}(t)] \\
&= \frac{C}{2}\langle F_n(t)|P_iP_i|F_n(t)\rangle \mathbf{d}_n(t) \\
&- \frac{C}{2}\sum_m \langle F_n(t)|L_{ij}|F_m(t)\rangle\langle F_m(t)|L_{ij}|F_n(t)\rangle \\
&\quad \times \mathbf{d}_m(t)\mathbf{d}_m(t)^\dagger \mathbf{d}_n(t),
\end{aligned}
\tag{4.9}
$$

where we omitted some self-contraction term.

5. Lorentz-invariant Action of **D** field and Hamilton's Principle

In this section, we propose Hamilton's principle of Lorentz-invariant action of **D** field and investigate what conditions or approximations are needed to reproduce from it Heisenberg's equation of motion of **D** field (4.9) given in non-covariant light cone frame. We imagine, from the explicit form of second-quantized Hamiltonian operator (4.5), that there exists Hamilton's variation principle based on the Lorentz invariant action I:

$$\delta I = 0 \tag{5.1}$$

with

$$I = \frac{1}{2}\{\mathrm{Tr}(\mathbf{D}^\dagger P_\mu \mathbf{D} P_\mu) + \mathrm{Tr}(P_\mu \mathbf{D}^\dagger P_\mu \mathbf{D})\} - \kappa\, \mathrm{Tr}\,(\mathbf{D}^\dagger L_{\mu\nu}\mathbf{D}\mathbf{D}^\dagger L_{\mu\nu}\mathbf{D}), \tag{5.2}$$

where Tr means trace with respect to Hilbert space I. One sees immediately that I possesses a wider class of invariances under transformations on Hilbert space I which include Lorentz transformation generated by $\exp(i\epsilon_{\mu\nu}L_{\mu\nu})$ with infinitesimal parameters $\epsilon_{\mu\nu}$ and $L_{\mu\nu} \equiv (L_{ij}, M_i)$ given in section 2.

Now the variation with respect to $\mathbf{D}^\dagger$ gives, by omitting some self-contraction term,

$$\delta_{\mathbf{D}^\dagger} I \equiv \mathrm{Tr}[(\delta\mathbf{D}^\dagger)(P_\mu \mathbf{D} P_\mu - 2\kappa L_{\mu\nu}\mathbf{D}\mathbf{D}^\dagger L_{\mu\nu}\mathbf{D})] = 0. \tag{5.3}$$

At this point, in order to make Euler-Lagrange's equation, which is derived from the above variation principle, to come near Heisenberg's equation (4.9) given in light cone frame, let us rewrite the above equation in the following form, separating the so-called transverse part from longitudinal part in terms of the light cone variables:

$$\begin{aligned}\mathrm{Tr}[\delta\mathbf{D}^\dagger\{&P_i\mathbf{D}P_i - \big(P^+\mathbf{D}P^- + P^-\mathbf{D}P^+\big)\\ &- 2\kappa L_{ij}\mathbf{D}\mathbf{D}^\dagger L_{ij}\mathbf{D}\\ &+ 4\kappa\big(L_{i+}\mathbf{D}\mathbf{D}^\dagger L_{i-}\mathbf{D} + L_{i-}\mathbf{D}\mathbf{D}^\dagger L_{i+}\mathbf{D}\big)\\ &- 4\kappa L_{+-}\mathbf{D}\mathbf{D}^\dagger L_{-+}\mathbf{D}\}] = 0,\end{aligned} \tag{5.4}$$

where indices i or j denote the transverse components running from 1 to 9 and $\pm$ the longitudinal components $\frac{1}{\sqrt{2}}(0 \pm 11)$ in 11 dimensional space-time.

Next let us rewrite the above equation in terms of light cone bases $|F_n\rangle's$, by assuming the existence of a particular solution $\mathbf{D}$ of the non-covariant form (4.2) according to the discussion given in section 3. In fact, we assume here the solution to satisfy

$$[X^+, \mathbf{D}] = \mathbf{0} \tag{5.5}$$

and further

$$P_i[P_i, \mathbf{D}] = 0,^\star \tag{5.6}$$

for a reason stated below, by taking into account the commutativity $[P_i, X^+] = 0$ under Yang's space-time algebra, which we preferably adopt in what follows.

$\star$ If one starts from beginning by restricting oneself to Yang's space-time algebra where the relation $[P_i^2, X^+] = 0$ is algebraically guaranteed, one can choose the light cone bases $|F_n(t) >' s$ introduced in section 3 as common eigenstates of X^+, P_i^2 and $\mathbf{D}$ defined by (4.2), so that there holds the relation $[P_i^2, \mathbf{D}] = 0$. Then one finds that, under the latter relation, the assumption $P_i[P_i, \mathbf{D}] = 0$ introduced here turns out to be equivalent to $[P_i, \mathbf{D}]P_i = 0$ or $[P_i, [P_i, \mathbf{D}]] = 0$.

In this case, under the variation

$$\delta \mathbf{D}^\dagger \equiv \sum_{t,n} |F_n(t)\rangle \delta \mathbf{d}_n^\dagger(t) \langle F_n(t)|, \tag{5.7}$$

(5.4) leads us to the following Euler-Lagrange's equation:

$$\begin{aligned}
&\langle F_n(t)|P_i P_i|F_n(t)\rangle \mathbf{d}_n(t) \\
&- \sum_{t',m} \{\langle F_n(t)|P^+|F_m(t')\rangle\langle F_m(t')|P^-|F_n(t)\rangle \\
&\qquad + \langle F_n(t)|P^-|F_m(t')\rangle\langle F_m(t')|P^+|F_n(t)\rangle\}\mathbf{d}_m(t') \\
&- 2\kappa \sum_m \langle F_n(t)|L_{ij}|F_m(t)\rangle\langle F_m(t)|L_{ij}|F_n(t)\rangle \mathbf{d}_m(t)\mathbf{d}_m^\dagger(t)\mathbf{d}_n(t) \\
&+ 4\kappa \sum_{t',m} \{\langle F_n(t)|L_{i+}|F_m(t')\rangle\langle F_m(t')|L_{i-}|F_n(t)\rangle \\
&\qquad + \langle F_n(t)|L_{i-}|F_m(t')\rangle\langle F_m(t')|L_{i+}|F_n(t)\rangle\}\mathbf{d}_m(t')\mathbf{d}_m^\dagger(t')\mathbf{d}_n(t) \\
&- 4\kappa \sum_{t',m} \langle F_n(t)|L_{+-}|F_m(t')\rangle\langle F_m(t')|L_{-+}|F_n(t)\rangle \mathbf{d}_m(t')\mathbf{d}_m^\dagger(t')\mathbf{d}_n(t) = 0.
\end{aligned} \tag{5.8}$$

In the above expression, one should notice that the first term on the left-hand side is well written in the present form on account of (5.6), so as to reproduce the corresponding term in (4.9). With respect to the third term, it is important to note that L_{ij} are commutative with $X^\pm$, $[L_{ij}, X^\pm] = 0$, so the intermediate states are all retained on light cone time t. Secondly, since $L_{i+}(= i[X_i, X^+])$ appearing in the fourth term is also commutative with X^+, i.e., $[L_{i+}, X^+] = 0$, one sees again that t' appearing in the intermediate states is set equal to t. Therefore, $\langle F_n(t)|L_{i+}|F_m(t)\rangle \equiv i\langle F_n(t)|[X_i, X^+]|F_m(t)\rangle = 0$, and thus the fourth term entirely disappears.

Before discussing the fifth term, let us consider the second term in (5.8), from which we expect to extract the most interesting term including the time derivatives of $\mathbf{d}_n(t)$ corresponding to the left-hand side of (4.9). At this point, if one adopts Yang's space-time algebra as mentioned above, one finds that $[P^+, X^+] = 0$,

$[P^-, X^+] = iN$ and $[N, X^+] = iP^+$. Consequently $\langle F_n(t)|P^+|F_m(t')\rangle$ in the term can be written as

$$\langle F_n(t)|P^+|F_m(t')\rangle \equiv P^+_{nm}(t,t') = P^+_{nm}(t)\delta(t-t'). \tag{5.9}$$

On the other hand, $\langle F_m(t')|P^-|F_n(t)\rangle$ is also written down, through the repeated use of commutation relations mentioned above, in the following form

$$\begin{aligned}\langle F_m(t')|P^-|F_n(t)\rangle &= \frac{1}{(t-t')}\langle F_m(t')|[P^-, X^+]|F_n(t)\rangle \\ = \frac{i}{(t-t')}\langle F_m(t')|N|F_n(t)\rangle &= \frac{-1}{(t-t')^2}\langle F_m(t')|P^+|F_n(t)\rangle.\end{aligned} \tag{5.10}$$

In this way one can rewrite the second term in (5.8) as

$$\begin{aligned}&-\sum_{t',m}\{\langle F_n(t)|P^+|F_m(t')\rangle\langle F_m(t')|P^-|F_n(t)\rangle \\ &\quad + \langle F_n(t)|P^-|F_m(t')\rangle\langle F_m(t')|P^+|F_n(t)\rangle\}\mathbf{d}_m(t') \\ &= 2\sum_{t',m}\frac{P^+_{nm}(t,t')P^+_{mn}(t',t)}{(t'-t)^2}\mathbf{d}_m(t') \\ &= -2\sum_{t',m}\frac{P^+_{nm}(t,t')}{(t'-t)}P^+_{mn}(t)\{\frac{d}{dt'}\delta(t-t')\}\mathbf{d}_m(t').\end{aligned} \tag{5.11}$$

One finds that the above singular expression includes the following term with first-order time derivative of $\mathbf{d}_n(t)$, in addition to second- and zeroth-order time-derivative terms, as

$$-2i\sum_m H_{nm}(t)\frac{d}{dt}\mathbf{d}_m(t), \tag{5.12}$$

where

$$H_{nm}(t) \equiv iP^+_{mn}(t)\big(\frac{d}{dt'}P^+_{nm}(t,t')\big)|_{t'=t}. \tag{5.13}$$

In combining the above results, (5.8) as a whole can be written in the following

form

$$
\begin{aligned}
&2i\sum_m H_{nm}(t)\frac{d}{dt}\mathbf{d}_m(t) \\
&= \langle F_n(t)|P_iP_i|F_n(t)\rangle \mathbf{d}_n(t) \\
&- 2\kappa\sum_m \langle F_n(t)|L_{ij}|F_m(t)\rangle\langle F_m(t)|L_{ij}|F_n(t)\rangle \mathbf{d}_m(t)\mathbf{d}_m(t)^\dagger \mathbf{d}_n(t),
\end{aligned} \tag{5.14}
$$

neglecting the second- and zeroth-order time derivative terms of $\mathbf{d}_n(t)$ mentioned above and one *anomalous* term which comes from the fifth term in (5.8) and will be discussed soon later. As a matter of fact, further neglecting the non-diagonal parts of H_{nm}, one finds that the above equation (5.14) reproduces ultimately (4.9) under the following correspondence

$$
\frac{\hbar}{C} \sim H_{nn}(t) \tag{5.15}
$$

and $\kappa = 1/2$. One might renormalize the n-dependent factor of $H_{nn}(t)$, if necessary, into $\mathbf{d}_n(t)$ and $\mathbf{d}_n^\dagger$, still keeping the commutation relations (4.3) and (4.4).

Finally let us explicitly calculate the *anomalous* term mentioned above, which comes from the fifth term on the left-hand side of (5.8):

$$
-4\kappa\sum_{t',m}\langle F_n(t)|L_{+-}|F_m(t')\rangle\langle F_m(t')|L_{-+}|F_n(t)\rangle \mathbf{d}_m(t')\mathbf{d}_m^\dagger(t')\mathbf{d}_n(t). \tag{5.16}
$$

By remarking

$$
[L_{+-}, X^\pm] = \mp iX^\pm, \tag{5.17}
$$

one finds

$$
\langle F_n(t)|L_{+-}|F_m(t')\rangle = it\frac{\delta(t-t')}{(t-t')}\delta_{mn}. \tag{5.18}
$$

Consequently, (5.16) can be written as

$$
-4\kappa\sum_{t'} A(t,t')\mathbf{d}_n(t')\mathbf{d}_n^\dagger(t')\mathbf{d}_n(t), \tag{5.19}
$$

where anomalous factor $A(t,t')$ is defined by

$$A(t,t') \equiv \frac{tt'}{(t-t')^2}\delta^2(t-t'). \tag{5.20}$$

One finds that this is a kind of self-interactions of D particles with anomalous nonlocal time factor $A(t,t')$.

At this point, it should be noticed that L_{+-} appearing in $\langle F_m(t')|L_{+-}|F_n(t)\rangle$ is the boost operator in the 11-th spatial direction $L_{0\ 11}$ and the light cone basis state $F_n(t)$ defined by (3.2) is understood as a limiting state attained at the maximal boost in the same direction. Consequently we conjecture that the origin of anomaly is in the overfull operation of boost on the light cone basis state.

6. Conclusions and Discussions

In this paper, we have proposed Hamilton's principle of relativistic action of matrix model, on the basis of a general framework describing nonlocal field associated with noncommutative space-time of Snyder-Yang's type. In section 4, we have found that both noncommutativities appearing in position coordinates of D particles in matrix model and in quantized space-time are eventually unified through second quantization of matrix model. In section 5, starting from the relativistic action principle of **D** field, we have tried to reproduce Heisenberg's equation of motion (4.9) derived in section 4 in close contact with Hamiltonian in matrix model and clarified what conditions or approximations are needed for the purpose. Anomalous aspects encountered there, in self-interaction or self-mass terms , for instance, seem to be deeply concerned with the limiting character of light cone bases or the infinite momentum frame, while we do not enter into detailed discussions on these problems in the present paper.

It should be noted here that space-time algebra of Yang' type has played an important role in reproducing Heisenberg's equation (4.9) from Hamilton's action principle mentioned above. In fact, in executing the latter process in section 5, we

have preferably adopted Yang's algebra, which has simple characteristic commutation relations of light cone time operator X^+ with $P^{\pm}$, compared with Snyder's one.

In connection with the above argument, it is interesting to ask how the space-time algebra is to be chosen, or how the structure of Hilbert space I and II is possibly related, although in the present paper space-time algebra $\mathcal{R}$ is regarded as one given *kinematically* from the beginning, as in the form of Snyder's type, Yang's type and so on. There exists, however, another interesting possibility that space-time algebra $\mathcal{R}$ is Lie algebra composed of Killing vector fields realized in the parameter space of Snyder-Yang's type, whose metric structure is determined *dynamically* in connection with dynamics of matter field in accordance with the thought of general theory of relativity.

This possibility may be actually formulated in applying Hamilton's principle to action I, by taking variations with respect to matrices of space-time quantities connected with Hilbert space I, in addition to the variation of $\mathbf{D}$ (5.7) connected with Hilbert space II. This approach seems to be attractive, if we consider that matrix model includes gravitation. It may add a new light to the realization of noncommutative version of general theory of relativity or extended Kaluza-Klein theory,[12] which possibly governs the world of Planck length.

Finally let us comment on the correspondence between the present nonlocal field theory based on quantized space-time and a conventional local field theory. As was mentioned at the end in section 2, the correspondence may be well given by means of quasi-local representation bases $|Q_n\rangle's$, where quasi-local field $\mathbf{U}$ may be expressed as $\mathbf{U} = \sum_n |Q_n\rangle \mathbf{u}_n \langle Q_n|$ analogously to (4.2). One can imagine that $\mathbf{u}_n ({\mathbf{u}_n}^\dagger)$ denotes annihilation (creation) operator of *excitation* state $|Q_n\rangle$ within quasi-local four-dimensional region (or "elementary domain" according to Yukawa[13]) and has correspondence to quantized local field $\mathbf{U}(x_\mu)$ with x_μ identified with expectation values of X_μ with respect to $|Q_n\rangle$ given in section 2. In the present paper, we have entirely omitted, for brevity of simplicity in expression, the

explicit use and discussions of physical dimensions, for instance, both of X_μ and P_μ being defined in (2.1) $\sim$ (2.4) as no-dimensional ones. This important problem, however, must be left to future discussions, which might clarify, with the aid of (5.15), the internal relation between Planck constant $\hbar$, Planck length and so on.

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